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On the Existence of Optimal Processes in Non-Stationary Environments

By

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We consider an aggregative model of intertemporal allocation under uncertainty, in which the utility and production functions are allowed to be time dependent, the random shocks occurring in each period are entirely arbitrary, and the production functions are permitted to be non-concave. In this framework, we provide a theorem on the existence of infinite-horizon optimal processes. In the course of establishing this result, we obtain the existence of optimal policy functions and we show that they are monotone in the stock levels.

1. Introduction

The main purpose of this paper is to provide a general theorem establishing the existence of optimal processes for an aggregative stochastic growth model. We also provide some monotonicity results for optimal processes in the context of this model. In our framework the utility and production functions are allowed to be time dependent, and the random shocks occurring in each period are entirely arbitrary, so, in particular, we do not assume that they be independent or identically distributed. We impose no concavity assumptions on the production functions but we do require the utility function at each date to be concave.

The existence theorem provided in this paper is one of the most

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general in the literature for the aggregative stochastic growth model considered here. Many authors have concentrated on the "stationary case," where the utility function at date t is given by $u_t(\cdot) = \delta^t u(\cdot)$ (where u is a period utility function, and $0 < \delta < 1$ is a discount factor), the production function is time invariant, and the random shocks to production are independent and identically distributed (see, for example, Brock and Mirman, 1972). The existence theorems for such models may be obtained by appealing to the dynamic programming arguments of Blackwell (1965) and Maitra (1968). The latter, however, for the aggregative stochastic growth model we consider, are special cases of the results proved in this paper.

For the nonstationary model, the closest existence theorem to the one presented in this paper is that of Bhattacharya and Majumdar (1981). Their result does allow for a more general set of models (that is, models other than the aggregative stochastic growth model); however, it depends critically on the convexity of the feasible set, which we do not require. Assumptions of convexity of feasible sets excludes a number of models that economists use and recently nonconvexities in feasible sets have received a great deal of attention, particularly in the natural resources literature (see, for example, Mirman and Spulber, 1982).

Another technique used in obtaining existence theorems is to endow the space containing the feasible set with an appropriate topology, and then to show that the feasible set is non-empty and compact and the objective function is upper-semicontinuous in that topology; and finally use the fact that an upper-semicontinuous function attains its maximum on a non-empty compact set. This approach has been used for the deterministic model by Majumdar (1975) and, for the stochastic model with a slightly different set of assumptions, by Chichilnisky (1981). It is possible to use such an approach for our model, too; however, there are a number of side-benefits from the method we use. In the course of our proof, we obtain the existence of optimal policy functions and we show that they are monotone. Further, our proof is constructive; we first show that finite-horizon optimal processes exist, then identify the limit of these processes as the optimal process for the infinite-horizon model.

The organization of this paper may be summarized as follows. In Section 2, we formally present the model and define feasible and optimal processes. In Section 3, we discuss the finite-horizon model. We prove that finite-horizon optimal processes exist, and that they may be obtained using optimal policy functions, which are functions of the beginning of period stock level and the partial history of the random shocks. We then indicate that these optimal policy functions are monotone in the beginning of period stock. In Proposition 3.1, we provide a set of assumptions under which the optimal policy functions are semi-Markovian. In Section 4, we discuss the infinite-horizon model. We show that the limit of the finite-horizon optimal processes is optimal for the infinite-horizon model, and that the optimal process may be obtained using optimal policy functions which are monotone. In Theorem 4.4, we provide a condition under which the optimal policy function is stationary. Proofs of all results can be found in Section 5. Section 6 contains a technical appendix.

1. The Model

a) The Environment

The environment is represented by the probability space (Ω, F, P) with the following interpretations:

(i) $\Omega = X_{t=0}^{\infty} \Omega_t$ where Ω_t is the set of possible states of the environment at date t; $r_t \in \Omega_t$ denotes the state at date t. We assume that Ω_t is a compact metric space and we let ϕ_t be its Borel field.

(ii) F is the sigma field on Ω induced by cylinder sets; that is, sets of the form $X_{t=0}^{\infty} A_t$ where $A_t \in \phi_t$ for all t and $A_t = \Omega_t$ for all but finitely many t.

(iii) P is the probability on (Ω, F) of the set of sequences of states of the environment.

We denote the partial history of the environment at date t ($t = 0, 1, \dots$) by $h_t = (r_0, r_1, \dots, r_t) \in X_{i=0}^{t} \Omega_i$. We denote by F_t the sub-sigma field of F induced by the partial history at date t; in particular, F_t is induced by cylinder sets of the form $X_{i=0}^{\infty} A_i$, where $A_i = \Omega_i$ for all $i > t$.

b) The Technology, Feasible Processes, and Policy Functions

The technology is represented by the sequence of production functions $\{f_t\}_{t=0}^{\infty}$ where for each t, $f_t : \mathbb{R}_+ \times \Omega_{t+1} \to \mathbb{R}_+$; if the investment at date t is x_t and the state of the environment at date $t+1$ is r_{t+1}, then the date $t+1$ output is $y_{t+1} = f_t(x_t, r_{t+1})$. We impose the following assumptions on the technology: for each $t = 0, 1, \dots$

(T.1) f_t is continuous on $\mathbb{R}_+ \times \Omega_{t+1}$.

(T.2) $f_t(x, r_{t+1})$ is nondecreasing in x for each fixed $r_{t+1} \in \Omega_{t+1}$.

Remark: For $t \geq 1$, r_t is important for two reasons. First, it determines the date t output from date $t-1$ input via the production function f_{t-1}; second, r_t may help predict the values of $\{r_{t+1}, r_{t+2}, \ldots\}$. The value of r_0, however, is important only in predicting values of $\{r_1, r_2, \ldots\}$. If the $\{r_t\}_{t=0}^{\infty}$ process were independent, then r_0 would be irrelevant.

The *pure accumulation process* from initial stock $y \geq 0$ at date τ, denoted by $\{k_t\}_{t=\tau}^{\infty}$, is defined inductively by

$$k_\tau = y \quad \text{and} \quad k_{t+1} = f_t(k_t, r_{t+1}) \qquad \text{for } t = \tau, \tau+1, \ \ldots \ . \tag{2.1}$$

For $t = 0, 1, \ldots$, define $\bar{f}_t : \mathbb{R}_+ \to \mathbb{R}_+$ by $\bar{f}_t(x) = \max\{f_t(x, r_{t+1}) : r_{t+1} \in \Omega_{t+1}\}$. Since f_t is continuous in its arguments and Ω_{t+1} is assumed compact, $\bar{f}_t$ is well-defined. The *pure accumulation sequence* from initial stock $y \geq 0$ at date τ, denoted by $\{\bar{k}_t\}_{t=\tau}^{\infty}$, is defined inductively by

$$\bar{k}_\tau = y \quad \text{and} \quad \bar{k}_{t+1} = \bar{f}_t(\bar{k}_t) \qquad \text{for } t = \tau, \tau+1, \ldots \ . \tag{2.2}$$

In specifying the finite-horizon model we require the tuple $e = (y, h_0, b, T)$ where $y \geq 0$ is the initial stock, $h_0 \in \Omega_0$ is the initial history, b is the (possibly random) terminal stock and $T = 0, 1, \ldots$ is the time horizon. We denote by M_T the set of non-negative F_T-measurable random variables; M_T is the set of possible *terminal stocks* for the T-horizon model.

Fix a time horizon $T = 0, 1, \ldots$, a date $t = 0, 1, \ldots, T$ and a terminal stock $b \in M_T$. We say that the terminal stock, b, can be *reached* from the initial stock-history pair $(y, h_t) \in \mathbb{R}_+ \times X_{i=0}^{t} \Omega_i$ at date t if, when we denote by $\{k_i\}_{i=t}^{\infty}$ the pure accumulation process from initial stock y at date t [defined in (2.1)], $P(k_T \geq b \mid h_t) = 1$. The tuple $e = (y, h_t, b, T)$ is a *date t admissible tuple* if the terminal stock, b, can be reached from the initial stock-history pair (y, h_t) at date t. A date 0 admissible tuple will be called simply an *admissible tuple*. Given any date t admissible tuple $e = (y, h_t, b, T)$ we define

$$\Gamma_t^T(y, h_t, b) = \{x \in [0, y] \mid b \text{ can be reached from } (x, h_t) \text{ at date } t\} \ . \tag{2.3}$$

Note that if (y, h_t, b, T) is a date t admissible tuple then $\Gamma_t^T(y, h_t, b)$ contains y and is therefore not empty.

Let $e = (y, h_\tau, b, T)$ be a date τ admissible tuple (where $T = 0, 1, \ldots$ and $\tau = 0, 1, \ldots, T$). The process $\{x_t, c_t, y_t\}_{t=\tau}^{T}$ is *e-feasible*, if for each $t = \tau+1, \ldots, T$,

$$x_t \text{ and } c_t \text{ are } F_t\text{-measurable random variables ;} \tag{2.4}$$

$$c_t + x_t = y_t = f_{t-1}(x_{t-1}, r_t);\ P(c_t \geq 0,\ x_t \geq 0 \mid h_\tau) = 1\ ; \tag{2.5}$$

$$y_\tau = y;\ c_\tau \in \mathbb{R}_+,\ x_\tau \in \mathbb{R}_+ \text{ and } c_\tau + x_\tau = y_\tau\ ;$$

and

$$P(x_T \geq b \mid h_\tau) = 1\ . \tag{2.6}$$

The process $\{x_t, c_t, y_t\}_{t=0}^{\infty}$ is *feasible for the infinite-horizon model* from the initial stock-history pair (y, h_τ) at date τ if (2.4) and (2.5) hold for each $t > \tau$. We shall refer to the processes $\{x_t\}$, $\{c_t\}$, and $\{y_t\}$ as the investment (or input), consumption, and output (or stock) processes, respectively.

If $\{x_t, c_t, y_t\}_{t=\tau}^{T}$ $(T \leq \infty)$ is any feasible process (for either the finite- or the infinite-horizon model) from the initial stock-history pair (y, h_τ) at date τ, then for each $t = \tau, \tau+1, \ldots, P(x_t \leq k_t \leq \bar{k}_t$, $c_t \leq k_t \leq \bar{k}_t$ and $y_t \leq k_t \leq \bar{k}_t \mid h_\tau) = 1$ where the processes $\{k_t\}_{t=\tau}^{\infty}$ and $\{\bar{k}_t\}_{t=\tau}^{\infty}$ are defined from initial stock y using (2.1) and (2.2) above, respectively.

Fix a $T = 0, 1, \ldots, \infty$, and a $\tau = 0, 1, \ldots, T$ (with $\tau < \infty$). The set of functions $\{\Pi_t\}_{t=\tau}^{T}$ where for each t, $\Pi_t : \mathbb{R}_+ \times X_{i=0}^{t}\, \Omega_i \to \mathbb{R}_+$ and $\Pi_t(y, \cdot) \leq y$ *generates* a unique process $\{x_t, c_t, y_t\}_{t=\tau}^{T}$ from the initial stock-history pair (y_τ, h_τ) at date τ as follows: set $x_\tau = \Pi_\tau(y_\tau, h_\tau)$, $c_\tau = y_\tau - x_\tau$ and given any (x_t, c_t, y_t) for $t \geq \tau$ set $y_{t+1} = f_t(x_t, r_{t+1})$, $x_{t+1} = \Pi_{t+1}(y_{t+1}, h_{t+1})$ and $c_{t+1} = y_{t+1} - x_{t+1}$.

Recall that if T is any finite horizon, M_T is the set of possible terminal stocks for the T-horizon model. For any date $t \leq T$ we define

$$D_t^T = \{(y, h_t, b) \in \mathbb{R}_+ \times X_{i=0}^{t}\, \Omega_i \times M_T \mid (y, h_t, b, T) \text{ is a date } t \text{ admissible tuple}\}\ . \tag{2.7}$$

Let $\{d_t^T\}_{t=0}^{T}$ be a set of functions such that $d_t^T : D_t^T \to \mathbb{R}_+$. We may consider d_t^T a function on $\mathbb{R}_+ \times X_{i=0}^{t}\, \Omega_i \times M_T$ by setting d_t^T equal to some arbitrary constant outside of D_t^T. The set of functions $\{d_t^T\}_{t=0}^{T}$ is a *set of policy functions* for the T-horizon model if for any admissible tuple $e = (y, h_0, b, T)$ the set of functions $\{d_t^T(\cdot, b)\}_{t=0}^{T}$ generates an e-feasible process from (y, h_0) at date 0.

c) Preferences, Optimal Processes and Optimal Policy Functions

Preferences are represented by the sequence of utility functions $\{u_t\}_{t=0}^{\infty}$ where for each t, $u_t : \mathbb{R}_+ \to \mathbb{R}_+$ and $u_t(c)$ is the utility obtained from the consumption of $c \geq 0$ at date t. We assume that for each $t = 0, 1, \ldots$

(U.1) u_t is continuous on $\mathbb{R}_+$;
(U.2) u_t is monotone nondecreasing on $\mathbb{R}_+$;
(U.3) u_t is concave on $\mathbb{R}_+$.

Let $e = (y, h_\tau, b, T)$ be a date τ admissible tuple. An e-feasible process $\{x_t^*, c_t^*, y_t^*\}_{t=\tau}^T$ is *e-optimal* if for every e-feasible process $\{x_t, c_t, y_t\}_{t=\tau}^T$,

$$E[\sum_{t=\tau}^{T} u_t(c_t^*) \mid h_\tau] \geq E[\sum_{t=\tau}^{T} u_t(c_t) \mid h_\tau] \ . \tag{2.8}$$

A process $\{x_t^*, c_t^*, y_t^*\}_{t=\tau}^\infty$ is *optimal* for the infinite-horizon model from the initial stock-history pair (y, h_τ) at date τ if it is feasible and if for any other process $\{x_t, c_t, y_t\}_{t=\tau}^\infty$ feasible from (y, h_τ) at date τ,

$$\limsup_{N \to \infty} E[\sum_{t=\tau}^{N} (u_t(c_t) - u_t(c_t^*)) \mid h_\tau] \leq 0 \ . \tag{2.9}$$

Recall that policy functions, and how they generate feasible processes, were discussed in Section 2.b). Fix a time horizon, $T < \infty$. The set of policy functions $\{d_t^T\}_{t=0}^T$ is a *set of optimal policy functions* if for each date τ admissible tuple $e = (y, h_\tau, b, T)$ $(\tau \leq T)$, the functions $\{d_t^T(\cdot, b)\}_{t=\tau}^T$ generate an e-optimal process from (y, h_τ) at date τ. For the infinite-horizon model a set of optimal policy functions, $\{d_t^\infty\}_{t=0}^\infty$, is analogously defined.

3. The Finite-Horizon Model

In this section we show that finite-horizon optimal processes and policy functions exist and that they may be chosen so as to have certain monotonicity properties.

Lemma 3.1: Fix a finite time horizon T. Then there exist functions $\{W_t^T\}_{t=0}^T$ and $\{g_t^T\}_{t=0}^T$, where $W_t^T : D_t^T \to \mathbb{R}_+$ and $g_t^T : D_t^T \to \mathbb{R}_+$ for $t = 0, \ldots, T$, such that

(i) $W_T^T(y, h_T, b) = \max_{x \in \Gamma_T^T(y,h_t,b)} u_T(y - x)$.

(ii) $W_t^T(y, h_t, b) =$
$\max_{x \in \Gamma_t^T(y,h_t,b)} \{u_t(y - x) + E[W_{t+1}^T(f_t(x, r_{t+1}), h_{t+1}, b) \mid h_t]\}$,
for $t = 0, \ldots, T - 1$.

(iii) $g_T^T(y, h_T, b)$ solves the maximization problem in (i) and is the infimum of the set of solutions to this problem.

(iv) For $t = 0, \ldots, T-1$, $g_t^T(y, h_t, b)$ solves the maximization problem in (ii) and is the infimum of the set of solutions to this problem.

Theorem 3.1 (Existence):
Fix a finite time horizon T. Then the functions $\{g_t^T\}_{t=0}^T$ obtained in Lemma 3.1 constitute a set of optimal policy functions. Thus, given any admissible tuple $e = (y, h_0, b, T)$, the process $\{x_t, c_t, y_t\}_{t=0}^T$ generated by the functions $\{g_t^T\}_{t=0}^T$ is e-optimal.

Theorem 3.2 (Monotonicity of Optimal Policy Functions):
Fix a finite time horizon, T, and a date $t \leq T$. Let $e = (y, h_t, b, T)$ and $e' = (y', h_t, b', T)$ be two date t admissible tuples such that $y \geq y'$ and $P(b \geq b' \mid h_t) = 1$. Then $g_t^T(y, h_t, b) \geq g_t^T(y', h_t, b')$, where $\{g_t^T\}_{t=0}^T$ is the set of functions obtained in Lemma 3.1.

Before stating the next results, we introduce a definition.

Let $e = (y, h_0, b, T)$ be an admissible tuple. If $\{x_t, c_t, y_t\}_{t=0}^T$ is an e-optimal process, and for every e-optimal process $\{x'_t, c'_t, y'_t\}_{t=0}^T$, we have $P(x_t \leq x'_t \mid h_0) = 1$ for $t = 0, 1, \ldots, T$, then $\{x_t, c_t, y_t\}_{t=0}^T$ is called the *SCI-e-optimal process*. (Here SCI denotes "smallest capital input".) If $\{x_t, c_t, y_t\}_{t=0}^T$ is the SCI-e-optimal process, and $\{g_t^T\}_{t=0}^T$ is a set of optimal policy functions which generates $\{x_t, c_t, y_t\}_{t=0}^T$ for every admissible tuple e, then $\{g_t^T\}_{t=0}^T$ is the set of *SCI-optimal policy functions.*

Corollary 3.1 (SCI Optimal Process and Policies):
Let $e = (y, h_0, b, T)$ be an admissible tuple and let $\{x_t, c_t, y_t\}_{t=0}^T$ be the e-optimal process generated by the functions $\{g_t^T\}_{t=0}^T$ obtained in Lemma 3.1. Then $\{x_t, c_t, y_t\}_{t=0}^T$ is the SCI-e-optimal process, and $\{g_t^T\}_{t=0}^T$ is the set of SCI-optimal policy functions.

An immediate consequence of Theorem 3.2 is the monotonicity of SCI-optimal processes with respect to the initial and terminal stocks.

Corollary 3.2 (Monotonicity of SCI Optimal Processes):
Let $e = (y, h_0, b, T)$ and $e' = (y', h_0, b', T)$ be two admissible tuples with $y \geq y'$ and $P(b \geq b' \mid h_0) = 1$. Let $\{x_t, c_t, y_t\}_{t=0}^{T}$ and $\{x'_t, c'_t, y'_t\}_{t=0}^{T}$ be the SCI-e- and SCI-e'-optimal processes, respectively. Then for each $t = 0, 1, \ldots, T$, $P(x_t \geq x' \mid h_0) = 1$.

We now obtain a monotonicity result for SCI-optimal processes as we vary the time-horizon.

Corollary 3.3 (Monotonicity of SCI-optimal processes in T):
Fix a $y \geq 0$ and a finite time horizon T and let $e = (y, h_0, 0, T)$ and $e' = (y, h_0, 0, T+1)$. Denote by $\{x_t, c_t, y_t\}_{t=0}^{T}$ and $\{x'_t, c'_t, y'_t\}_{t=0}^{T+1}$ the SCI-e- and SCI-e'-optimal processes, respectively. Then for each $t = 0, \ldots, T$, $P(x_t \leq x'_t \mid h_0) = 1$.

At any date t, the history h_t plays two roles in the finite-horizon model: first, it helps predict $\{r_{t+1}, r_{t+2}, \ldots\}$, and second it may, in part, determine what the terminal stock will be at the last period, date T. The following condition denies h_t such roles.

Condition I: The process $\{r_t\}_{t=0}^{T}$ is independent (but not necessarily identically distributed) and the terminal stock b is a constant (that is, a degenerate random).

In Proposition 3.1 below, we show that under Condition I, SCI-optimal policy functions are independent of history h_t. Note, however, that SCI-optimal policy functions may still be dependent on the date t. A set of policy functions $\{d_t^T\}_{t=0}^{T}$ shall be called *Semi-Markov* if for each t, $d_t^T(y, h_t, b)$ is independent of the history h_t.

Proposition 3.1 (Semi Markov Optimal Policy Functions):
Under condition I, the set of SCI-optimal policy functions is Semi-Markov.

4. The Infinite-Horizon Model

In this section we show that the limit of SCI-$(y, h_0, 0, T)$ optimal processes, as the horizon T tends to infinity, is an optimal process for the infinite-horizon model under a joint restriction on the technology and preferences (see condition E below). The optimal process may be generated by a set of optimal policy functions which have a monotonicity property (see Theorem 4.2 below). Under stationarity assumptions on the $\{r_t\}_{t=0}^{\infty}$ process and the production and utility functions (see condition S below), we show in Theorem 4.4 that these optimal policy functions are stationary (that is, independent of the history and the date).

Fix an initial stock-history pair (y, h_0) and for each $T = 0, 1, \ldots$, let $e^T = (y, h_0, 0, T)$ and let $\{x_t^T, c_t^T, y_t^T\}_{t=0}^{T}$ be the SCI-e^T-optimal process. The *limit process* from (y, h_0), denoted by $\{\tilde{x}_t, \tilde{c}_t, \tilde{y}_t\}_{t=0}^{T}$, is defined by setting $\tilde{y}_0 = y$, $\tilde{x}_t = \lim_{T\to\infty} x_t^T$ for all t, and defining $\{\tilde{c}_t, \tilde{y}_t\}_{t=0}^{\infty}$ from $\{\tilde{x}_t\}_{t=0}^{\infty}$ in the obvious manner [that is, using (2.5) above]. Let $\{\bar{k}_t\}_{t=0}^{\infty}$ be the pure accumulation sequence from initial stock y at date 0 [defined via (2.2)]. Then from Corollary 3.3, for each T and t, $x_t^T \leq x_t^{T+1} \leq \bar{k}_t$ so $\tilde{x}_t = \lim_{T\to\infty} x_t^T$ is well-defined.

Recall that we obtained SCI-optimal policy functions, $\{g_t^T\}_{t=0}^{T}$, in Corollary 3.1. Define the functions $\{g_t^{\infty}\}_{t=0}^{\infty}$ where for each t, $g_t^{\infty} : \mathbb{R}_+ \times X_{i=0}^{t}\Omega_i \to \mathbb{R}_+$ by $g_t^{\infty}(y, h_t) = \lim_{T\to\infty} g_t^T(y, h_t, 0)$; it is clear that the set of functions $\{g_t^{\infty}\}_{t=0}^{\infty}$ generate the limit process, and they will be called the *limit SCI policy functions.*

Condition E: Given any initial stock-history pair (y, h_0), there exists a process $\{x_t', c_t', y_t'\}_{t=0}^{\infty}$ feasible from (y, h_0) such that if $\{k_t\}_{t=0}^{\infty}$ is the pure accumulation process [defined in (2.1)] from the initial stock y at date 0, then

$$\lim_{N\to\infty} E[\sum_{t=0}^{N}(u_t(k_t) - u_t(c_t')) \mid h_0] < \infty \ .$$

Theorem 4.1 (Existence):

Fix an initial stock-history pair (y, h_0). Under condition E the limit process from (y, h_0) is optimal from (y, h_0) for the infinite-horizon model and may be generated by the limit SCI policy functions, $\{g_t^{\infty}\}_{t=0}^{\infty}$.

Remarks:

(i) Condition E holds when the utility function is of the form $u_t(\cdot) = \delta^t u(\cdot)$ with $0 \leq \delta < 1$ and either $u(\cdot)$ is bounded above, or there exists a maximum sustainable stock (that is, there exists $K > 0$ such that for all t, $f_t(x, r_{t+1}) < x$ for all $x \geq K$ and $r_{t+1} \in \Omega_{t+1}$). The latter is used in Brock and Mirman (1972).

(ii) In the existence theorem of Bhattacharya and Majumdar (1981) the assumption $\sum_{t=0}^{\infty} u_t(Ek_t) < \infty$ is used, which ensures Condition E above, since the concavity of the utility function implies $Eu_t(k_t) < u_t(Ek_t)$ for all t, by Jensen's inequality. However, there are models where $E\sum_{t=0}^{\infty} u_t(k_t) = \infty$ (and hence $\sum_{t=0}^{\infty} u_t(Ek_t) = \infty$) and yet Condition E holds. For example, let $f_t(x, r_{t+1}) = \rho x$ for some $\rho > 1$, $u_t(c) = c/(1+c)$ and $y = 1$. Then $k_t = \rho^t$, and $\rho^t/(1+\rho^t) \to 1$ as $t \to \infty$ so $\sum_{t=0}^{\infty} u_t(k_t) = \sum_{t=0}^{\infty} \rho^t/(1+\rho^t) = \infty$. However, we may consider a process $\{x_t, c_t, y_t\}_{t=0}^{\infty}$ inductively by defining $a = (1/2)[1 + (1/\rho)]$, $y_0 = 1$, and for $t \geq 0$, $y_{t+1} = \rho a y_t$, $x_t = a y_t$, $c_t = (1-a)y_t$. Since $0 < a < 1$, it is clear that $\{x_t, c_t, y_t\}_{t=0}^{\infty}$ is feasible from $y = 1$. Note that for each t, $y_t = (\rho a)^t$ so $c_t = (1-a)(\rho a)^t$ and therefore

$$\sum_t [u_t(k_t) - u_t(c_t)] \leq \sum_t \left[1 - \frac{c_t}{1+c_t}\right] \leq \sum_t \frac{1}{c_t} = \frac{1}{1-a}\sum_t \frac{1}{(\rho a)^t} . \tag{4.1}$$

Since $(\rho a) > 1$ the summation on the right-hand side of (4.1) is finite so condition E holds.

(iii) Without Condition E there may not exist an optimal process. As an example one may consider the "cake-eating" model of Gale (1967). Let $f_t(x, r_{t+1}) = x$, $u_t(c) = c/(1+c)$ and $y = 1$. Then $k_t = 1$ for all t, and

$$\sum_{t=0}^{\infty}[u_t(k_t) - u_t(c_t)] = \sum_{t=0}^{\infty}[(1/2) - \{c_t/(1+c_t)\}] ,$$

so if Condition E holds, $[(1/2) - \{c_t/(1+c_t)\}] \to 0$ and $c_t \to 1$ as $t \to \infty$; hence, using $c_t \leq y_t = x_{t-1} \leq 1$, we obtain $x_t \to 1$ as $t \to \infty$. But then $c_t = y_t - x_t = x_{t-1} - x_t \to 0$ as $t \to \infty$, a contradiction to $c_t \to 1$ as $t \to \infty$ so condition E does not hold. Also there is no optimal process, for if $\{x_t^*, c_t^*, y_t^*\}_{t=0}^{\infty}$ is optimal, then for some τ, $c_\tau^* > 0$ and one may construct an alternative process $\{x_t, c_t, y_t\}_{t=0}^{\infty}$ as follows: $(x_t, y_t, c_t) = (x_t^*, c_t^*, y_t^*)$ for $t < \tau$ and $t > \tau + 1$; $y_\tau = y_\tau^*$, $c_\tau = c_{\tau+1} = (c_\tau^* + c_{\tau+1}^*)/2$, $x_\tau = y_{\tau+1} = y_\tau - c_\tau$, $x_{\tau+1} = x_{\tau+1}^*$. One may check that $\{x_t, c_t, y_t\}_{t=0}^{\infty}$ is feasible. Using the strict concavity of the utility function, one may check that the process $\{x_t, c_t, y_t\}_{t=0}^{\infty}$ dominates $\{x_t^*, c_t^*, y_t^*\}_{t=0}^{\infty}$ [that is, (2.9) fails] which contradicts the optimality of $\{x_t^*, c_t^*, y_t^*\}_{t=0}^{\infty}$.

We now state a monotonicity result for the infinite-horizon optimal processes and policy functions obtained in Theorem 4.1. This is an immediate consequence of Theorem 3.2 and Corollary 3.2.

Theorem 4.2 (Monotonicity of Limit SCI Policy Functions and Processes):
Fix a y and y' with $y \geq y' \geq 0$. Then for each date $t = 0, 1, \ldots$, and partial history h_t, $g_t^\infty(y, h_t) \geq g_t^\infty(y', h_t)$. Further, if $\{x_t, c_t, y_t\}_{t=0}^\infty$ and $\{x_t', c_t', y_t'\}_{t=0}^\infty$ are the limit SCI processes from initial stock-history (y, h_0) and (y', h_0), respectively, then $P(x_t \geq x_t' \mid h_0) = 1$ for each t.

Remark:
A similar monotonicity result has been obtained by Dechert and Nishimura (1983) for the deterministic and stationary model (that is, where condition S below holds), and by Majumdar, Mitra, and Nyarko (1989) for the stochastic and stationary model.

Recall that we showed in Proposition 3.1 that if the $\{r_t\}_{t=0}^\infty$ process is independent (but not necessarily identically distributed), and the terminal stock is a constant, then the SCI-optimal policy functions, $\{g_t^T\}_{t=0}^T$, are Semi-Markov (that is, independent of the history, h_t). From the definition of g_t^∞, we therefore obtain the following result.

Theorem 4.3 (Semi-Markov Policy Functions):
Suppose that the process $\{r_t\}_{t=0}^\infty$ is independent (but not necessarily identically distributed). Then the limit SCI policy functions $\{g_t^\infty\}_{t=0}^\infty$ are Semi-Markov; that is, the functions $g_t^\infty(y, h_t)$ are independent of the history, h_t.

Condition S: The process $\{r_t\}_{t=0}^\infty$ is independent and identically distributed; the utility functions are of the form $u_t(c) = \delta^t u(c)$, where $0 \leq \delta < 1$; and $f_t = f$ for all t.

We now show that under condition S, the limit SCI policy functions, $\{g_t^\infty\}_{t=0}^\infty$, are independent of both the history, h_t, and the date, t. Following the terminology of Blackwell (1965) and Maitra (1968), we call such policy functions *stationary (or Markov).*

Theorem 4.4 (Stationary Optimal Policies):
Under condition S, the limit SCI policy functions, $\{g_t^\infty\}_{t=0}^\infty$, are

stationary. That is, the functions $g_t^\infty(y, h_t)$ are independent of both the partial history, h_t, and the date t, and we may write for each t, $g_t^\infty(y, h_t) = g^\infty(y)$ for some $g^\infty : \mathrm{I\!R}_+ \to \mathrm{I\!R}_+$.

5. Proofs

We provide, in this section, the proofs of the results stated in Sections 3 and 4.

Proof of Lemma 3.1:

Fix a $T = 0, 1, \ldots$ and recall that for $t = 0, 1, \ldots, T$, Γ_t^T and D_t^T are defined in (2.3) and (2.7) above, respectively. We define the functions $\{W_t^T\}_{t=0}^T$ where $W_t^T : D_t^T \to \mathrm{I\!R}_+$ by backward induction as follows:

$$W_T^T(y, h_T, b) = \max_{x \in \Gamma_T^T(y, h_T, b)} u_T(y - x) \tag{5.1}$$

and given W_{t+1}^T for any $t = 0, 1, \ldots, T-1$,

$$W_t^T(y, h_t, b) = \max_{x \in \Gamma_t^T(y, h_t, b)} \{u_t(y - x) + E[W_{t+1}^T(f_t(x, r_{t+1}), h_{t+1}, b) \mid h_t]\} . \tag{5.2}$$

Note that if $x \in \Gamma_t^T(y, h_t, b)$ then $P[(f_t(x, r_{t+1}), h_{t+1}, b) \in D_{t+1}^T \mid h_t] = 1$ so the maximization in (5.2) is well defined.

In the Claim below we show that solutions to (5.1) and (5.2) exist. Denote by $g_t^T(y, h_t, b)$ the infimum of the set of solutions to (5.2) [or (5.1) if $t = T$]; the Claim shows that g_t^T is a solution that satisfies certain measurability properties. Recall that F_t is the sigma field generated by the partial history h_t; we shall sometimes write F_t-measurable random variables as $y(h_t)$ to emphasize this fact (see Chung, 1974, Lemma 9.1.2, p. 279, for more on this).

Claim: Fix a finite-horizon, T, and a $t = 0, \ldots, T$. Then

(a) there exists a function $g_t^T : D_t^T \to \mathrm{I\!R}_+$ such that for all $(y, h_t, b) \in D_t^T$, $g_t^T(y, h_t, b)$ is the infimum of the set of solutions, and is itself a solution, to the maximization problem defining $W_t^T(y, h_t, b)$.

(b) $W_t^T(y, h_t, b)$ is continuous in y for fixed $(h_t, b) \in X_{i=0}^T \Omega_i \times M_T$.

(c) Fix a terminal stock $b \in M_T$. If $y(h_t)$ is any uniformly bounded F_t-measurable random variable such that $(y(h_t), h_t, b) \in D_t^T$ for all $h_t \in X_{i=0}^t \Omega_i$ then $g_t^T(y(h_t), h_t, b)$ and $W_t^T(y(h_t), h_t, b)$ are F_t-measurable random variables.

Proof of the Claim: It is useful to consider the terminal stock, b, as fixed throughout this proof. The proof amounts to checking, by backward induction, that conditions of the Measurable Selection Theorem of Furukawa hold (see Lemma A.1 of the appendix); this will prove parts of (a) and (c) of the claim. We then prove (b) by invoking the Maximum Theorem (see Berge, 1963, p. 116).

For any $Y > 0$ and $t = 0, 1, \ldots, T$, let $D_t^{T,Y} = \{(y, h_t, b) \in D_t^T \mid y \leq Y\}$. It is clear that if for any fixed t, the claim holds when we restrict the domain of W_t^T to $D_t^{T,Y}$, with Y arbitrary, then the claim holds when the domain of W_t^T is D_t^T.

We seek to use Lemma A.1 in the appendix. First, consider the case where $t = T$. To apply Lemma A.1, fix a $Y > 0$ and set $S = X_{i=0}^T \Omega_i$ and $A = [0, Y]$. Let $y(h_t)$ be as in part (c) of the Claim [which clearly includes the case where $y(h_t)$ is some constant y] with $y(h_T) \leq Y$ for all h_T. Then for $s = h_T \in S$ set $A(s) = \Gamma_T^T(y(h_T), h_T, b)$, and for $(s, a) = (h_T, x) \in S \times A$ set $H(s, a) = u_T(y(h_T) - x)$.

Given these definitions we now verify that conditions (i)–(iv) of Lemma A.1 hold: (i) follows from Lemmas A.2 and A.3 (b) of the appendix; (ii) follows from the F_T-measurability of $y(h_T)$ and the continuity (hence, measurability) of u_T; (iii) follows from continuity of u_T; and (iv) follows from the fact that $y(h_T)$ and x are restricted to lie in $[0, Y]$ and the fact that u_T is continuous.

Hence, we may apply Lemma A.1 to show that for $t = T$, (a) and (c) of the Claim hold. To show that (b) holds observe that the objective function $u_T(y - x)$ is continuous in (y, x); also from Lemma A.3 (a) of the appendix, the constraint set $\Gamma_T^T(y, h_T, b)$ is a continuous correspondence in y. Part (b) of the claim then follows from the Maximum Theorem (see Berge, 1963, p. 116). Hence, the claim holds for $t = T$ when the domain of definition of W_T^T is $D_T^{T,Y}$; since $Y > 0$ is arbitrary, the claim holds for $t = T$.

Next let $\tau = 0, 1, \ldots, T-1$ and assume that the claim holds for $t = \tau + 1$. We proceed to show that the claim holds for $t = \tau$. The proof is very similar to that above for $t = T$, where now in applying Lemma A.1 we set, for fixed $Y > 0$, $A = [0, Y]$, $S = X_{i=0}^T \Omega_i$ and given $y(h_\tau)$ as in part (c) of the claim with $y(h_\tau) \leq Y$ for all h_τ, and $s = h_\tau \in S$, set $A(s) = \Gamma_\tau^T(y(h_\tau), h_\tau, b)$; and for $(s, a) = (h_\tau, x) \in S \times A$ set $H(s, a) = u_\tau(y(h_\tau) - x) + E[W_{\tau+1}^T(f_\tau(x, r_{\tau+1}), h_{\tau+1}, b) \mid h_\tau]$. One now uses the induction hypothesis (that is, the fact that the claim holds for $t = \tau + 1$) to mimic the steps used in proving the claim for $t = T$ above to show that Lemma A.1 holds and that the claim holds for $t = \tau$.

Q.E.D.

Proof of Theorem 3.1:

The theorem is established by verifying two claims.

Claim 1: The functions $\{g_t^T\}_{t=0}^T$ obtained in Lemma 3.1 constitute a set of policy functions.

Proof of Claim 1: Let $e = (y, h_0, b, T)$ be an admissible tuple and let $\{x_t, c_t, y_t\}_{t=0}^T$ be the process generated by the functions $\{g_t^T(\cdot, b)\}_{t=0}^T$ from (y, h_0) at date 0. We need to show that $\{x_t, c_t, y_t\}_{t=0}^T$ is e-feasible; that is, (2.4)–(2.6) hold.

Now (x_0, c_0, y_0) may be considered constants (given a h_0); if for $t = 0, 1, \ldots$, (x_t, c_t, y_t) are F_t-measurable then $y_{t+1} = f_t(x_t, r_{t+1})$ is F_{t+1}-measurable (from the continuity of f_t) so from part (c) of the claim established in the proof of Lemma 3.1, $x_{t+1} = g_t^T(y_{t+1}, h_{t+1}, b)$ (and hence $c_{t+1} = y_{t+1} - x_{t+1}$) is F_{t+1}-measurable. Therefore, (2.4) holds for all t. Next, to show (2.5) we need show only that for all t, $P(c_t = y_t - x_t \geq 0 \mid h_0) = 1$, the rest following by construction; but this follows from the definition of Γ_t^T and the fact that $P(x_t \in \Gamma_t^T(y_t, h_t, b) \mid h_0) = 1$. Finally, (2.6) follows from $P(x_T \in \Gamma_T^T(y_T, h_T, b) \mid h_0) = 1$. This establishes Claim 1.

We now show, in Claim 2 below, that the functions $\{g_t^T\}_{t=0}^T$ are a set of optimal policy functions. Further, the claim justifies the interpretation of $W_t^T(y, h_t, b)$ obtained in Lemma 3.1 as the value of continuing the T-horizon model with terminal stock b from the initial stock-history pair (y, h_t) at date t.

Claim 2: Let $e = (y, h_\tau, b, T)$ be a date τ admissible tuple and let $\{x_t, c_t, y_t\}_{t=\tau}^T$ be the process generated by the policy functions $\{g_t^T\}_{t=\tau}^T$ from the initial stock-history pair (y, h_τ) at date τ. If $\{x'_t, c'_t, y'_t\}_{t=\tau}^T$ is any other e-feasible process then

$$E[\sum_{t=\tau}^{T} u_t(c_t) \mid h_\tau] = W_\tau^T(y, h_\tau, b) \geq E[\sum_{t=\tau}^{T} u_t(c'_t) \mid h_\tau] \,. \tag{5.3}$$

In particular, $\{g_t^T\}_{t=0}^T$ is a set of optimal policy functions.

Proof of Claim 2: Let $e = (y, h_\tau, b, T)$, $\{x_t, c_t, y_t\}_{t=\tau}^T$ and $\{x'_t, c'_t, y'_t\}_{t=\tau}^T$ be as in the claim. Since for each $t = \tau, \dots, T$, x'_t is in the feasible set of the maximization exercise defining $W_t^T(y'_t, h_t, b)$ [see (5.1) and (5.2)] we obtain after taking expectations conditional on h_τ,

$$E[W_T^T(y'_T, h_T, b) \mid h_\tau] \geq E[u_T(c'_T) \mid h_\tau] \tag{5.4}$$

and for $t = \tau, \dots, T-1$

$$E[W_t^T(y'_t, h_t, b) \mid h_\tau] \geq E[u_t(c'_t) \mid h_\tau] + E[W_{t+1}^T(y'_{t+1}, h_{t+1}, b) \mid h_\tau]\ . \tag{5.5}$$

Adding (5.4) and (5.5) (over $t = \tau, \dots, T-1$) and rearranging,

$$W_\tau^T(y'_\tau, h_\tau, b) \geq E[\sum_{t=\tau}^T u_t(c'_t) \mid h_\tau]\ . \tag{5.6}$$

Next, since for each t, x_t is a solution to the maximization exercise defining $W_t^T(y_t, h_t, b)$ we obtain equations similar to (5.4) and (5.5) with $\{x_t, c_t, y_t\}_{t=\tau}^T$ replacing $\{x'_t, c'_t, y'_t\}_{t=\tau}^T$ and equalities replacing inequalities. We can therefore show that

$$W_\tau^T(y_\tau, h_\tau, b) = E[\sum_{t=\tau}^T u_t(c_t) \mid h_\tau]\ . \tag{5.7}$$

But since $y'_\tau = y_\tau = y$, (5.3) follows from (5.6) and (5.7).

Q.E.D.

Proof of Theorem 3.2:

Let T, $e = (y, h_t, b, T)$ and $e' = (y', h_t, b', T)$ be as in Theorem 3.2. We shall prove the theorem by backward induction on the date, t. Since $g_T^T(y, h_T, b) = b \geq b' = g_T^T(y', h_T, b')$ (recall g_T^T is the smallest optimal investment at date T) the theorem holds for $t = T$. Suppose now that for some $\tau = 0, \dots, T-1$, the theorem holds for all $t = \tau+1, \dots, T$; we proceed to show that the theorem then holds for $t = \tau$.

To this end, suppose, per absurdum, that $x = g_\tau^T(y, h_\tau, b) < g_\tau^T(y', h_\tau, b') = x'$. We will show that $x \in \Gamma_\tau^T(y', h_\tau, b')$ and that

x is a solution to the maximization exercise defining $W_\tau^T(y', h_\tau, b')$; that is,

$$W_\tau^T(y', h_\tau, b') = u_\tau(y' - x) + E[W_{\tau+1}(f_\tau(x, r_{\tau+1}), h_{\tau+1}, b') \mid h_\tau] \,. \tag{5.8}$$

However, x' is by definition the smallest solution to the maximization exercise defining $W_\tau^T(y', h_\tau, b')$ while (5.8) implies that x is also a solution; this contradicts $x < x'$ and would complete the proof of the theorem.

We proceed to prove (5.8). Let $\{x_t, c_t, y_t\}_{t=\tau}^T$ [resp. $\{x'_t, c'_t, y'_t\}_{t=\tau}^T$] be the process generated by the policy functions $\{g_t^T(\cdot, b)\}_{t=\tau}^T$ [resp. $\{g_t^T(\cdot, b')\}_{t=\tau}^T$] from the initial stock-history pair (y, h_τ) [resp. (y', h_τ)] at date τ. We use these processes to construct two more processes $\{\hat{x}_t, \hat{c}_t, \hat{y}_t\}_{t=\tau}^T$ and $\{\hat{x}'_t, \hat{c}'_t, \hat{y}'_t\}_{t=\tau}^T$ as follows. Let $A_t = \{x_t < x'_t\}$ and let A_t^c be the complement of A_t. Define $\hat{y}_\tau = y_\tau$; $\hat{y}'_\tau = y'_\tau$; $\hat{x}_t = x'_t$ on A_t and $\hat{x}_t = x_t$ on A_t^c; $\hat{x}'_t = x_t$ on A_t and $\hat{x}'_t = x'_t$ on A_t^c; $\{\hat{c}_t, \hat{y}_t\}_{t=\tau}^T$ and $\{\hat{c}'_t, \hat{y}'_t\}_{t=\tau}^T$ are defined from $\{\hat{x}_t\}_{t=\tau}^T$ and $\{\hat{x}'_t\}_{t=\tau}^T$, respectively, in the obvious way [that is, using (2.5)]. We continue via a sequence of claims which we prove later.

Claim 1: For all $t > \tau$, $A_{t-1}^c A_t = \phi$.

Claim 2: The process $\{\hat{x}_t, \hat{c}_t, \hat{y}_t\}_{t=\tau}^T$ is (y, h_τ, b, T)-feasible.

Claim 3: The process $\{\hat{x}'_t, \hat{c}'_t, \hat{y}'_t\}_{t=\tau}^T$ is (y', h_τ, b', T)-feasible.

Remark: Since $\hat{x}'_\tau = x_\tau = x$, claim 3 implies that $x \in \Gamma_\tau^T(y', h_\tau, b')$.

Claim 4:

$$W_\tau^T(y', h_\tau, b') = u_\tau(\hat{c}'_\tau) + E[\sum_{t=\tau+1}^{T} u_t(\hat{c}'_t) \mid h_\tau] \,. \tag{5.9}$$

Claim 5: Equation (5.8) holds.

Proof of Claim 1: Let $t > \tau$. On A_{t-1}^c, $x_{t-1} \geq x'_{t-1}$; hence from the monotonicity of f_{t-1} in x, $y_t = f_{t-1}(x_{t-1}, r_t) \geq f_{t-1}(x'_{t-1}, r_t) = y'_t$; but then from the induction hypothesis $x_t = g_t^T(y_t, h_t, b) \geq g_t^T(y'_t, h_t, b') = x'_t$ which occurs on A_t^c. Hence, A_{t-1}^c is a subset of A_t^c, from which the claim follows.

Proof of Claim 2: We need to check that (2.4)–(2.6) hold for $\{\hat{x}_t, \hat{c}_t, \hat{y}_t\}_{t=\tau}^T$. Clearly, (2.4) follows from the fact that $A_t \in F_t$ and the F_t-measurability of x_t and x'_t. By construction, $\hat{c}_\tau = y_\tau - x'_\tau \geq y'_\tau - x'_\tau = c'_\tau \geq 0$; and for $\tau < t \leq T$, on $A_{t-1}A_t$, $\hat{c}_t = f_{t-1}(x'_{t-1}, r_t) - x'_t = c'_t \geq 0$; on $A^c_{t-1}A^c_t$, $\hat{c}_t = f_{t-1}(x_{t-1}, r_t) - x_t = c_t \geq 0$; on $A_{t-1}A^c_t$, $\hat{c}_t = f_{t-1}(x'_{t-1}, r_t) - x_t \geq f_{t-1}(x_{t-1}, r_t) - x_t = c_t \geq 0$; and $A^c_{t-1}A_t = \phi$ from Claim 1. Thus, for each $t = \tau, \ldots, T$, $P(\hat{c}_t \geq 0 \mid h_\tau) = 1$ and hence (2.5) holds. Finally, since $x_T = b \geq b' = x'_T$, $P(A_T) = 0$ so $P(\hat{x}_T = x_T \geq b \mid h_\tau) = 1$ and (2.6) holds.

Proof of Claim 3: This is very similar to the proof of Claim 2, so we omit the details.

Proof of Claim 4: Suppose, per absurdum, that (5.9) does not hold. Then from Claim 2 of Theorem 3.1 it must be the case that

$$E[\sum_{t=\tau}^{T} u_t(c'_t) \mid h_\tau] = W^T_\tau(y', h_\tau, b') > E[\sum_{t=\tau}^{T} u_t(\hat{c}'_t) \mid h_\tau] \,. \tag{5.10}$$

Again using Claim 2 of Theorem 3.1, we obtain

$$E[\sum_{t=\tau}^{T} u_t(c_t) \mid h_\tau] = W^T_\tau(y, h_\tau, b) \geq E[\sum_{t=\tau}^{T} u_t(\hat{c}_t) \mid h_\tau] \,. \tag{5.11}$$

So adding (5.10) and (5.11) and rearranging terms,

$$E[\sum_{t=\tau}^{T}(u_t(c_t) + u_t(c'_t) - u_t(\hat{c}_t) - u_t(\hat{c}'_t))h_\tau] > 0 \,. \tag{5.12}$$

We proceed to obtain a contradiction to (5.12). Define $L = \hat{c}_\tau = y_\tau - x'_\tau$, $M = c'_\tau = y'_\tau - x'_\tau$ and $\epsilon = x'_\tau - x_\tau$. Since $y_\tau \geq y'_\tau$ and $x'_\tau > x_\tau$ by assumption, we have $L \geq M$ and $\epsilon > 0$. Then, since u_τ is assumed concave,

$$\begin{aligned} u_\tau(c_\tau) + u_\tau(c'_\tau) - u_\tau(\hat{c}_\tau) - u_\tau(\hat{c}'_\tau) \; &= \\ [u_\tau(L+\epsilon) - u_\tau(L)] - [u_\tau(M+\epsilon) - u_\tau(M)] &\leq 0 \,. \end{aligned} \tag{5.13}$$

Next, for $\tau < t \leq T$, since $A^c_{t-1}A_t = \phi$ (and hence $\Omega = A_{t-1}A_t \cup A^c_{t-1}A^c_t \cup A_{t-1}A^c_t$), one can check that (using integrals to denote expectations conditional on h_τ),

$$\int u_t(\hat{c}_t) = \int_{A_{t-1}A_t} u_t(c'_t) + \int_{A^c_{t-1}A^c_t} u_t(c_t) + \int_{A_{t-1}A^c_t} u_t(\hat{c}_t) \quad (5.14)$$

and

$$\int u_t(\hat{c}'_t) = \int_{A_{t-1}A_t} u_t(c_t) + \int_{A^c_{t-1}A^c_t} u_t(c'_t) + \int_{A_{t-1}A^c_t} u_t(\hat{c}'_t)\ . \quad (5.15)$$

Using (5.14) and (5.15), and noting that $\Omega - A_{t-1}A_t - A^c_{t-1}A^c_t = A_{t-1}A^c_t$, one can verify that

$$\begin{aligned} &\int \{u_t(c_t) + u_t(c'_t) - u_t(\hat{c}_t) - u_t(\hat{c}'_t)\} \\ &= \int_{A_{t-1}A^c_t} \{u_t(c_t) + u_t(c'_t) - u_t(\hat{c}_t) - u_t(\hat{c}'_t)\} \\ &= \int_{A_{t-1}A^c_t} [u_t(L+\epsilon) - u_t(L)] - [u_t(M+\epsilon) - u_t(M)]\ , \end{aligned} \quad (5.16)$$

where $L = \hat{c}_t = f_{t-1}(x'_{t-1}, r_t) - x_t$, $M = c_t = f_{t-1}(x_{t-1}, r_t) - x_t$ and $\epsilon = x_t - x'_t$. On $A_{t-1}A^c_t$, $x'_{t-1} > x_{t-1}$ and $x_t \geq x'_t$, so $L \geq M$ and $\epsilon \geq 0$. Since u_t is concave the right-hand side of (5.16) is nonpositive, hence

$$\int \{u_t(c_t) + u_t(c'_t) - u_t(\hat{c}_t) - u_t(\hat{c}'_t)\} \leq 0\ . \quad (5.17)$$

From (5.13) and (5.17) [recalling that the integrals in (5.17) are conditional on h_τ], we obtain a contradiction to (5.12). This completes the proof of Claim 4.

Proof of Claim 5: From Claim 4 above

$$W^T_\tau(y', h_\tau, b') = u_\tau(\hat{c}'_\tau) + E[E[\sum_{t=\tau+1}^{T} u_t(\hat{c}'_t) \mid h_{\tau+1}] \mid h_\tau]]\ ,$$

and using Claim 2 of Theorem 3.1,

$$W^T_\tau(y', h_\tau, b') \leq u_\tau(\hat{c}'_\tau) + E[W^T_{\tau+1}(\hat{y}'_{\tau+1}, h_{\tau+1}, b') \mid h_\tau]\ . \quad (5.18)$$

Combining (5.18) with (5.2) yields (5.8). Q.E.D.

Proof of Corollary 3.1:

Suppose the corollary is false and let $m = \min\{t = 0, \ldots, T \mid P(x_t > x'_t \mid h_0) > 0\}$ and $B = \{x_m > x'_m\}$; then $P(B \mid h_0) > 0$. If $m = 0$ then $y_m = y'_m = y$; otherwise $m - 1 \geq 0$ and $P(x_{m-1} \leq x'_{m-1} \mid h_0) = 1$ so $P(y_m \leq y'_m \mid h_0) = 1$. In either case if $A = \{y_m \leq y'_m\}$ then $P(A \mid h_0) = 1$. Using the monotonicity result of Theorem 3.2 we may conclude that on AB, $x'_m < x_m \equiv g^T_m(y_m, h_m, b) \leq g^T_m(y'_m, h_m, b)$. Since $g^T_m(y'_m, h_m, b)$ is the smallest solution to the maximization exercise defining $W^T_m(y'_m, h_m, b)$ in (5.1) and (5.2), x'_m is therefore *not* a solution; we may therefore mimic the proof of Claim 2 of Theorem 3.1 and show that (5.5) with $\tau = 0$ and $t = m$ [or (5.4) if $m = T$] holds with strict inequality on AB. Hence, upon integration we may show that (5.6) holds with strict inequality, and therefore combining with (5.7) we obtain a contradiction to the optimality of $\{x'_t, c'_t, y'_t\}^T_{t=0}$. Q.E.D.

Proof of Corollary 3.3:

Define $e'' = (y, h_0, x'_T, T)$ where x'_T is the date T investment of the SCI-e'-optimal process. From the Principle of Optimality (see Lemma A.4 in the appendix), $\{x'_t, c'_t, y'_t\}^T_{t=0}$ is e''-optimal. It is easy to see that $\{x'_t, c'_t, y'_t\}^T_{t=0}$ is the SCI-e''-optimal process. Since $P(x'_T \geq 0 \mid h_0) = 1$ we may apply Corollary 3.2 to the SCI-e- and -e''-optimal processes $\{x_t, c_t, y_t\}^T_{t=0}$ and $\{x'_t, c'_t, y'_t\}^T_{t=0}$, respectively, to obtain $P(x_t \leq x'_t \mid h_0) = 1$ for each t. Q.E.D.

Proof of Proposition 3.1:

The function $g^T_t(y, h_t, b)$ is the solution to the maximization exercise in (5.1) and (5.2). If $\{r_t\}^T_{t=0}$ is independent then the objective function on the right-hand side of (5.2) involves an unconditional expectation and is therefore independent of the partial history, h_t; if, in addition, the terminal stock, b, is a constant then the constraint set $\Gamma^T_t(y, h_t, b)$ is independent of the partial history h_t. Hence, $g^T_t(y, h_t, b)$ is independent of h_t. Q.E.D.

Proof of Theorem 4.1:

Fix an initial stock-history pair $(y, h_0) \in \mathbb{R}_+ \times \Omega_0$; we shall use E_0 to denote expectations conditional on h_0. First, we show that the limit process is feasible. For fixed t, x_t^T is F_t-measurable, so (2.4) holds. Next, since f_{t-1} is continuous, $\tilde{c}_t = f_{t-1}(\tilde{x}_{t-1}, r_t) - \tilde{x}_t = \lim_{T\to\infty} f_{t-1}(x_{t-1}^T, r_t) - x_t^T = \lim_{T\to\infty} c_t^T \geq 0$; it is therefore clear that all the conditions in (2.5) hold. Hence, the limit process is feasible.

Let $\{k_t\}_{t=0}^{\infty}$ be the pure accumulation process from initial stock y [defined in (2.1)] at date 0, and define

$$\alpha = \inf \lim_{N\to\infty} E_0 \sum_{t=0}^{N} [u_t(k_t) - u_t(c_t)] \ , \tag{5.19}$$

where the infimum is taken over all processes $\{x_t, c_t, y_t\}_{t=0}^{\infty}$ that are feasible from (y, h_0). Note that the limit in (5.19) is either infinite or convergent since for all t, $P[u_t(k_t) - u_t(c_t) \geq 0 \mid h_0] = 1$. From Condition (A.1), α is finite. We now show

Claim 1: The limit process attains the infimum in (5.19).

Proof of Claim 1: Suppose, per absurdum, that the limit process does not attain the infimum in (5.19); then for some $\epsilon > 0$,

$$\lim_{N\to\infty} E_0 \sum_{t=0}^{N} [u_t(k_t) - u_t(\tilde{c}_t)] \geq \alpha + \epsilon \ , \tag{5.20}$$

so for some $N' > 0$,

$$E_0 \sum_{t=0}^{N'} [u_t(k_t) - u_t(\tilde{c}_t)] \geq \alpha + \frac{\epsilon}{2} \ . \tag{5.21}$$

Recall that for each $T = 0, 1, \ldots$, $\{x_t^T, c_t^T, y_t^T\}_{t=0}^{T}$ is the SCI-$(y, h_0, 0, T)$-optimal process. Since $\lim_{T\to\infty} c_t^T = \tilde{c}_t$, we may use the Dominated Convergence Theorem (since $c_t^T \leq k_t$ for each T) to show, using (5.21), that for some $T' \geq N'$,

$$E_0 \sum_{t=0}^{N'} [u_t(k_t) - u_t(c_t^T)] \geq \alpha + \frac{\epsilon}{4} \text{ for all } T \geq T' \ (\geq N') \ . \tag{5.22}$$

As $E_0[u_t(k_t) - u_t(c_t^T)] \geq 0$ for each t and T, (5.22) implies

$$E_0 \sum_{t=0}^{T} [u_t(k_t) - u_t(c_t^T)] \geq \alpha + \frac{\epsilon}{4} \text{ for all } T \geq T' \ . \tag{5.23}$$

Next, from the definition of α as an infimum we may choose a sequence of processes $\{\bar{x}_t^T, \bar{c}_t^T, \bar{y}_t^T\}_{t=0}^{\infty}$, $T = 1, 2, \ldots$, each feasible from (y, h_0) for the infinite-horizon model, such that for each T,

$$\lim_{N \to \infty} E_0 \sum_{t=0}^{N} [u_t(k_t) - u_t(\bar{c}_t^T)] \leq \alpha + \frac{\epsilon}{T} . \tag{5.24}$$

But for all t and T, $E_0[u_t(k_t) - u_t(\bar{c}_t^T)] \geq 0$, hence from (5.24),

$$E_0 \sum_{t=0}^{T} [u_t(k_t) - u_t(\bar{c}_t^T)] \leq \alpha + \frac{\epsilon}{T} \text{ for all } T . \tag{5.25}$$

It is clear that $\{\bar{x}_t^T, \bar{c}_t^T, \bar{y}_t^T\}_{t=0}^{T}$ is $(y, h_0, 0, T)$-feasible; thus, since $\{x_t^T, c_t^T, y_t^T\}_{t=0}^{T}$ is $(y, h_0, 0, T)$-optimal,

$$E_0 \sum_{t=0}^{T} [u_t(k_t) - u_t(c_t^T)] \leq \alpha + \frac{\epsilon}{T} \text{ for all } T . \tag{5.26}$$

Finally (5.23) and (5.26) imply that for $T \geq T'$, $\alpha + (\epsilon/4) \leq \alpha + (\epsilon/T)$; so taking limits as $T \to \infty$ and using $\epsilon > 0$ leads to a contradiction which proves Claim 1.

To complete the proof of Theorem 4.1 we need to show that the limit process is optimal; suppose, per absurdum, that this is not the case and, in particular, there is a process $\{\hat{x}_t, \hat{c}_t, \hat{y}_t\}_{t=0}^{\infty}$ feasible from the initial stock-history pair (y, h_0) such that for some $N'' < \infty$ and some $J > 0$, we have for all $N \geq N''$

$$E_0 \sum_{t=0}^{N} [u_t(\hat{c}_t) - u_t(\tilde{c}_t)] \geq J , \tag{5.27}$$

so

$$0 < J \leq E_0 \sum_{t=0}^{N} [u_t(k_t) - u_t(\tilde{c}_t)] - E_0 \sum_{t=0}^{N} [u_t(k_t) - u_t(\hat{c}_t)] . \tag{5.28}$$

Taking limits as $N \to \infty$ in (5.28) and using Claim 1,

$$0 < J \leq \alpha - \lim_{N \to \infty} E_0 \sum_{t=0}^{N} [u_t(k_t) - u_t(\hat{c}_t)] . \tag{5.29}$$

But from the definition of α the limit in (5.29) is no less than α; so (5.29) implies $0 < J \leq 0$, a contradiction. Thus the limit process is optimal. Q.E.D.

Proof of Theorem 4.4:

We shall consider all finite-horizon models below as having terminal stock, $b = 0$; so from Proposition 3.1, we may write $g_t^T(y, h_t, b) = g_t^T(y)$; we may also, for a similar reason, write $W_t^T(y, h_t, b) = W_t^T(y)$ where $W_t^T(y)$ is defined in (5.1) and (5.2).

Fix a finite-horizon, T. We will first show by backward induction on the date, t, that for $0 \leq t \leq T$,

$$\delta W_t^T(y) = W_{t+1}^{T+1}(y) \,. \tag{5.30}$$

First, $\delta W_T^T(y) = \delta \cdot \delta^T u(y) = W_{T+1}^{T+1}(y)$, so (5.30) holds for $t = T$. Next, suppose that (5.30) holds for some $t = \tau$ with $0 < \tau \leq T$; then, using (5.2) and the induction hypothesis,

$$\begin{aligned} \delta W_{\tau-1}^T(y) &= \delta[\max_{x\in[0,y]}\{\delta^{\tau-1}u(y-x) + EW_\tau^T(f(x,r))\}] && (5.31)\\ &= \max_{x\in[0,y]}\{\delta^\tau u(y-x) + EW_{\tau+1}^{T+1}(f(x,r))\} && (5.32)\\ &= W_\tau^{T+1}(y) && (5.33)\end{aligned}$$

(where the expectation is over r, which has the common distribution of the $\{r_t\}$ process, which is assumed i. i. d.). This shows that (5.30) holds for $t = \tau - 1$. Hence, by induction, (5.30) holds for all $t = 0, \ldots, T$.

Since, from (5.30), the expressions to be maximized in (5.31) and (5.32) differ by only a constant factor, δ, their sets of solutions must be identical; but notice that $g_{\tau-1}^T(y)$ [resp., $g_\tau^{T+1}(y)$] is the supremum of the set of solutions to the maximization in (5.31) [resp. (5.32)], hence $g_{\tau-1}^T(y) = g_\tau^{T+1}(y)$ for each $\tau = 1, \ldots, T$; taking limits as $T \to \infty$ results in

$$g_{\tau-1}^\infty(y) = \lim_{T\to\infty} g_{\tau-1}^T(y) = \lim_{T\to\infty} g_\tau^{T+1}(y) = g_\tau^\infty(y) \tag{5.34}$$

from which the theorem follows. Q.E.D.

6. Appendix

We now state some results that were used in the proofs. Let A be a compact subset of a metric space with metric μ, and define 2^A to be the set of all non-empty closed subsets of A. Given any two sets C and D in 2^A, we define the *Hausdorff metric*, h, on 2^A, by

$$h(C, D) = \max\{\sup_{c \in C} \mu(c, D), \ \sup_{d \in D}(d, C)\} ,$$

where $\mu(c, D) = \inf_{d \in D} \mu(c, d)$ and $\mu(d, C) = \inf_{c \in C} \mu(d, c)$.

The Borel field on 2^A, denoted by $B(2^A)$, is the smallest sigma field on 2^A containing all sets open with respect to h. Let S be a Borel subset of a complete and separable metric space and let $B(S)$ be its Borel sigma field. A correspondence $F : S \to 2^A$ is measurable [and we use the notation $F \in B(S)/B(2^A)$], if for each set $M \in B(2^A)$, $F^{-1}(M) = \{s : F(s) \cap M \text{ is non-empty}\} \in B(S)$. We may now state an important selection theorem.

Lemma A.1 (Measurable Selection Theorem):
Let S be a Borel subset of a complete and separable metric space. Suppose $H : S \times A \to \mathbb{R}^1$ and
(i) A is a compact subset of $\mathbb{R}^1$; $A(\cdot)$ is a correspondence such that for all $s \in S$, $A(s) \in 2^A$ and $A(\cdot) \in B(S)/B(2^A)$;
(ii) $H(s, a)$ is Borel measurable in s, for each fixed a in A;
(iii) $H(s, a)$ is continuous in a, for each fixed s in S;
(iv) $H(s, a)$ is uniformly bounded in (s, a).
Then (a) $M(s) = \max\{H(s, a) : a \in A(s)\}$ is Borel measurable; (b) there exists a Borel measurable function, $g : S \to A$, such that for all s in S, $g(s) \in A(s)$ and $H(s, g(s)) = \max\{H(s, a) : a \in A(s)\}$; (c) the function g above may be chosen so that $g(s) = \min\{a' \in A(s) : H(s, a') = \max_{a \in A(s)} H(s, a)\}$.

Proof: See Furukawa (1972, Theorem 4.1, p. 1619). Note that the selection used in that paper is the lexicographical maximum, which on the real line reduces to the simple maximum of a set of real numbers. It is clear that one could very easily replace the lexicographic maximum with the lexicographic minimum in the Furukawa selection result, from which (c) above follows. Q.E.D.

Recall that the correspondence Γ_t^T was defined in (2.3) with domain of definition D_t^T, the set of date t admissible tuples; also recall that

M_T is the set of possible terminal stocks for the T-horizon model [see Section 2.b)].

Lemma A.2: Fix a finite time horizon T, a $t = 0, \ldots, T$, and a terminal stock $b \in M_T$. There exists an F_t-measurable random variable, $b_t(h_t)$, such that for $(y, h_t, b) \in D_t^T$, $\Gamma_t^T(y, h_t, b) = \{x \mid b(h_t) \leq x \leq y\}$.

Proof: We shall prove the lemma by backward induction on the date, t. Since b is F_T-measurable we may write $b = b(h_T)$; hence $\Gamma_T^T(y, h_T, b) = \{x \mid b(h_T) \leq x \leq y\}$ so the lemma holds for $t = T$.

Next suppose the lemma holds for some date $t = \tau + 1$ where $0 \leq \tau \leq T$; we proceed to show that it then holds for $t = \tau$. Define

$$b_\tau(h_\tau) = \inf\{z \geq 0 \mid P[f_\tau(z, r_{\tau+1}) \geq b_{\tau+1}(h_{\tau+1}) \mid h_\tau] = 1\} \ , \quad (6.1)$$

where $b_{\tau+1}$ is obtained from the induction hypothesis. It is clear that $\Gamma_\tau^T(y, h_\tau, b) = \{x \mid b_\tau(h_\tau) \leq x \leq y\}$; it remains only to show that $b_\tau(h_\tau)$ is F_τ-measurable. However, for each $a \geq 0$,

$$b_\tau^{-1}(a, \infty) = \{h_\tau \in X_{i=0}^{\tau}\Omega_i \mid P[f_\tau(a, r_{\tau+1}) \geq b_{\tau+1}(h_{\tau+1}) \mid h_\tau] = 1 \ . \quad (6.2)$$

Since the conditional probability in (6.2) is an F_τ-measurable random variable, $b_\tau^{-1}([a, \infty)) \in F_\tau$ so $b_\tau(h_\tau)$ is F_τ-measurable.

Q.E.D.

Lemma A.3: Let T, t, and b be as in Lemma A.2. Then

(a) the correspondence $\Gamma_t^T(y, h_t, b)$ is continuous in y for fixed $h_t \in X_{i=0}^{t}\Omega_i$;

(b) for each F_t-measurable random variable $y(h_t)$, taking values in some compact interval A of $\mathbb{R}_+$ such that $(y(h_t), h_t, b) \in D_t^T$ for all $h_t \in X_{i=0}^{t}\Omega_i$, $\Gamma_t^T(y(h_t), h_t, b) \in F_t/B(2^A)$.

Proof: From Lemma A.2 it is easy to show that (a) holds. Next, note that from Lemma A.2, if $y(h_t)$ is as in (b) above,

$$\Gamma_t^T(y(h_t), h_t, b) = \Gamma_{t,1}^T(y(h_t), h_t, b) \cap \Gamma_{t,2}^T(y(h_t), h_t, b) \ , \quad (6.3)$$

where $\Gamma_{t,1}^T(y(h_t), h_t, b) = \{x \mid b(h_t) \leq x\}$ and $\Gamma_{t,2}^T(y(h_t), h_t, b) = \{x \mid 0 \leq x \leq y(h_t)\}$. To prove part (b) of the lemma it suffices to show that $\Gamma_{t,1}^T$ and $\Gamma_{t,2}^T$ are F_t-measurable since the intersection of two measurable correspondences is a measurable correspondence (see Rockafellar, 1969, Corollary 1.3, p. 9).

Let C be any closed and bounded subset of $\mathbb{R}_+$ and let $\bar{c} = \sup C$ and $\underline{c} = \inf C$. Then

$$\begin{aligned}(\Gamma_{t,1}^T)^{-1}(C) &= \{h_t : \Gamma_{t,1}^T(y(h_t), h_t, b) \cap C \text{ is non-empty}\} \\ &= \{h_t : \bar{c} \geq b_t(h_t)\} \ , \qquad (6.4)\end{aligned}$$

$$\begin{aligned}(\Gamma_{t,2}^T)^{-1}(C) &= \{h_t : \Gamma_{t,2}^T(y(h_t), h_t, b) \cap C \text{ is non-empty}\} \\ &= \{h_t : \underline{c} \leq y(h_t)\} \ . \qquad (6.5)\end{aligned}$$

As $b_t(h_t)$ and $y(h_t)$ are both F_t-measurable, the inverses in (6.4) and (6.5) both belong to F_t. If C is not a closed set one or both of the weak inequalities in (6.4) and (6.5) become strict inequalities; but even in this case the inverses still belong to F_t. Since $y(h_t)$ is assumed uniformly bounded it suffices to consider C bounded. Hence, $\Gamma_{t,1}^T$ and $\Gamma_{t,2}^T$, and therefore their intersection, Γ_t^T, are F_t-measurable.

Q.E.D.

Lemma A.4 (Principle of Optimality):
Let $e = (y, h_0, b, T+1)$ be an admissible tuple and let $\{x_t, c_t, y_t\}_{t=0}^{T+1}$ be an e-optimal process. Then $\{x_t, c_t, y_t\}_{t=0}^{T}$ is (y, h_0, x_T, T)-optimal.

Proof: Suppose, per absurdum, that the lemma is false. Then there exists a (y, h_0, x_T, T)-feasible process, $\{\hat{x}_t, \hat{c}_t, \hat{y}_t\}_{t=0}^{T}$ such that

$$E[\sum_{t=0}^{T} u_t(\hat{c}_t) \mid h_0] > E[\sum_{t=0}^{T} u_t(c_t) \mid h_0] \ . \qquad (6.6)$$

Construct a process $\{x_t', c_t', y_t'\}_{t=0}^{T+1}$ from (y, h_0) as follows: set $y_0' = y$, $x_t' = \hat{x}_t$ for $0 \leq t \leq T$ and $x_{T+1}' = b$; then define the $\{c_t', y_t'\}_{t=0}^{T+1}$ processes from $\{x_t'\}_{t=0}^{T+1}$ in the obvious manner [that is, using (2.5)]. Since $x_T' = \hat{x}_T \geq x_T$, $y_{T+1}' \geq y_{T+1} \geq b = x_{T+1}'$ and therefore one may check that $\{x_t', c_t', y_t'\}_{t=0}^{T+1}$ is $(y, h_0, b, T+1)$-feasible. Since $c_{T+1}' \geq c_{T+1}$, we may use (6.6) to show that

$$E[\sum_{t=0}^{T+1} u_t(c_t') \mid h_0] > E[\sum_{t=0}^{T+1} u_t(c_t) \mid h_0] \ , \qquad (6.7)$$

which contradicts the optimality of $\{x_t, c_t, y_t\}_{t=0}^{T+1}$. Q.E.D.

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